SHARPENING AND GENERALIZATIONS OF CARLSON'S DOUBLE INEQUALITY FOR THE ARC COSINE FUNCTION

BAI-NI GUO AND FENG QI

ABSTRACT. In this paper, we sharpen and generalize Carlson's double inequality for the arc cosine function.

1. Introduction and main results

In [1, p. 700, (1.14)] and [2, p. 246, 3.4.30], it was listed that

$$\frac{6(1-x)^{1/2}}{2\sqrt{2}+(1+x)^{1/2}}<\arccos x<\frac{\sqrt[3]{4}\,(1-x)^{1/2}}{(1+x)^{1/6}},\quad 0\le x<1. \tag{1}$$

The first aim of this paper is to sharpen and generalize the right-hand side inequality in (1) as follows.

Theorem 1. *For real numbers a and b, let*

$$f_{a,b}(x)=\frac{(1+x)^b}{(1-x)^a}\arccos x,\quad x\in(0,1). \tag{2}$$

(1) *If and only if*

$$(a,b)\in\left\{b\le\frac{2}{\pi}-a\right\}\cap\left\{a\le\frac{1}{2}\right\}, \tag{3}$$

the function $f_{a,b}(x)$ is strictly decreasing;

(2) *If*

$$(a,b)\in\left\{\frac{2}{\pi}-a\le b\le a-\frac{4}{\pi^2}\right\}\cup\left\{\frac{1}{2}\le a\le b+\frac{1}{3}\right\} \tag{4}$$

$$\cup\left\{\frac{1}{3}<a-b<\frac{4}{\pi^2},a+b\ge\frac{2(a-b)^{3/2}}{\sqrt{4(a-b)-1}}\right\}, \tag{5}$$

the function $f_{a,b}(x)$ is strictly increasing;

(3) *If*

$$(a,b)\in\left\{\frac{1}{3}<a-b<\frac{4}{\pi^2}\right\}\cap\left\{\frac{2}{\pi}-b<a\le\frac{1}{2}\right\}, \tag{6}$$

the function $f_{a,b}(x)$ has a unique maximum;

(4) *If*

$$(a,b)\in\left\{\frac{1}{3}<a-b<\frac{4}{\pi^2}\right\}\cap\left\{\frac{1}{2}<a\le\frac{2}{\pi}-b\right\}, \tag{7}$$

the function $f_{a,b}(x)$ has a unique minimum;

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(5) *If*

$$(a,b)\in\left\{\frac{1}{3}<a-b<\frac{4}{\pi^2}\right\}\cap\left\{\frac{2}{\pi}<a+b<\frac{2(a-b)^{3/2}}{\sqrt{4(a-b)-1}}\right\}\cap\left\{a>\frac{1}{2}\right\},\tag{8}$$

the function $f_{a,b}(x)$ has a unique maximum and a unique minimum in sequence;

(6) *The necessary condition for the function $f_{a,b}(x)$ to be strictly increasing is*

$$(a,b)\in\left\{b\ge\frac{2}{\pi}-a\right\}\cap\left\{a\ge\frac{1}{2}\right\}.\tag{9}$$

As direct consequences of the monotonicity of the function $f_{a,b}(x)$, the following inequalities may be deduced.

Theorem 2. *For $x\in(0,1)$, the double inequality*

$$\frac{\pi}{2}\cdot\frac{(1-x)^{1/2}}{(1+x)^b}<\arccos x<2^{b+1/2}\cdot\frac{(1-x)^{1/2}}{(1+x)^b}\tag{10}$$

holds provided that $b\ge\frac{1}{6}$.

The right-hand side inequality in (10) is valid if and only if $b\ge\frac{1}{6}$.

The reversed version of (10) is valid provided that $b\le\frac{2}{\pi}-\frac{1}{2}$.

The reversed version of the left-hand side inequality in (10) is valid if and only if $b\le\frac{2}{\pi}-\frac{1}{2}$.

If (a,b) satisfies (6), $16ab(b-a)+(a+b)^2>0$ and

$$x_1=\frac{(a+b)(2b-2a+1)-\sqrt{16ab(b-a)+(a+b)^2}}{2(a-b)^2}>0,\tag{11}$$

then

$$\left.\begin{matrix}\min\left\{2^{b+1/2},\dfrac{\pi}{2}\right\}\dfrac{(1-x)^a}{(1+x)^b}, & a=\dfrac{1}{2}\\ 0, & a<\dfrac{1}{2}\end{matrix}\right\}\le\arccos x$$

$$\le\frac{(1+x_1)^{b+1/2}(1-x_1)^{1/2-a}}{a+b+(a-b)x_1}\cdot\frac{(1-x)^a}{(1+x)^b}.\tag{12}$$

If (a,b) satisfies (7), $16ab(b-a)+(a+b)^2>0$ and

$$x_2=\frac{(a+b)(2b-2a+1)+\sqrt{16ab(b-a)+(a+b)^2}}{2(a-b)^2}\in(0,1),\tag{13}$$

then

$$\arccos x\ge\frac{(1+x_2)^{b+1/2}(1-x_2)^{1/2-a}}{a+b+(a-b)x_2}\cdot\frac{(1-x)^a}{(1+x)^b}.\tag{14}$$

The second aim of this paper is to sharpen and generalize the left-hand side inequality in (1) as follows.

Theorem 3. *For $x\in(0,1)$, the function*

$$F_{1/2,1/2,2\sqrt2}(x)=\frac{2\sqrt2+(1+x)^{1/2}}{(1-x)^{1/2}}\arccos x\tag{15}$$

is strictly decreasing. Consequently, the double inequality

$$\frac{6(1-x)^{1/2}}{2\sqrt2+(1+x)^{1/2}}<\arccos x<\frac{\left(1/2+\sqrt2\right)\pi(1-x)^{1/2}}{2\sqrt2+(1+x)^{1/2}}\tag{16}$$

holds on $(0,1)$ and the constants 6 and $\left(\frac{1}{2}+\sqrt2\right)\pi$ in (16) are the best possible.

Remark 1. From Theorem 2, we obtain the following two double inequalities:

$$\frac{\pi(1-x)^{1/2}}{2(1+x)^{1/6}} < \arccos x < \frac{\sqrt[3]{4}\,(1-x)^{1/2}}{(1+x)^{1/6}}, \quad x \in (0,1); \tag{17}$$

$$\frac{4^{1/\pi}(1-x)^{1/2}}{(1+x)^{(4-\pi)/2\pi}} < \arccos x < \frac{\pi(1-x)^{1/2}}{2(1+x)^{(4-\pi)/2\pi}}, \quad x \in (0,1). \tag{18}$$

Except that the right-hand side inequality in (17) and the left-hand side inequality in (16) are same to the corresponding one in (1) and that the left-hand side inequality in (1) is better than the corresponding one in (18), other corresponding inequalities in (1), (16), (17) and (18) are not included each other.

2. Proofs of theorems

Now we are in a position to verify our theorems.

Proof of Theorem 1. Straightforward differentiation yields

$$\begin{aligned} f'_{a,b}(x) &= \frac{(1+x)^{b-1}}{(1-x)^{a+1}}(\arccos x)\left[a+b+(a-b)x-\frac{\sqrt{1-x^2}}{\arccos x}\right] \\ &\triangleq \frac{(1+x)^{b-1}}{(1-x)^{a+1}}(\arccos x)g_{a,b}(x), \end{aligned} \tag{19}$$

$$\begin{aligned} g'_{a,b}(x) &= a-b-\frac{1}{(\arccos x)^2}+\frac{x}{\sqrt{1-x^2}\,\arccos x}, \\ g''_{a,b}(x) &= \frac{(\arccos x)^2+x\sqrt{1-x^2}\,\arccos x+2x^2-2}{(1-x^2)^{3/2}(\arccos x)^3} \\ &\triangleq \frac{h(x)}{(1-x^2)^{3/2}(\arccos x)^3}, \\ h'(x) &= \frac{(1+2x^2)}{\sqrt{1-x^2}}\left[\frac{3x\sqrt{1-x^2}}{1+2x^2}-\arccos x\right] \\ &\triangleq \frac{(1+2x^2)}{\sqrt{1-x^2}}q(x), \\ q'(x) &= \frac{4(1-x^2)^{3/2}}{(1+2x^2)^2}. \end{aligned}$$

It is clear that $q'(x)$ is positive, and so $q(x)$ is increasing on $[0,1)$. By virtue of $q(1)=0$, we obtain that $q(x)<0$ on $[0,1)$, which equivalent to $h'(x)<0$ and $h(x)$ is decreasing on $[0,1)$. Due to $h(1)=0$, it follows that $h(x)>0$ and $g''_{a,b}(x)>0$ on $[0,1)$, and so the function $g'_{a,b}(x)$ is increasing on $[0,1)$. It is easy to obtain that $\lim_{x\to0^+} g'_{a,b}(x)=a-b-\frac{4}{\pi^2}$ and $\lim_{x\to1^-} g'_{a,b}(x)=a-b-\frac{1}{3}$. Hence,

(1) if $a-b\ge\frac{4}{\pi^2}$, then $g'_{a,b}(x)>0$ and $g_{a,b}(x)$ is increasing on $(0,1)$;
(2) if $a-b\le\frac{1}{3}$, then $g'_{a,b}(x)<0$ and $g_{a,b}(x)$ is decreasing on $(0,1)$;
(3) if $\frac{1}{3}<a-b<\frac{4}{\pi^2}$, then $g'_{a,b}(x)$ has a unique zero and $g_{a,b}(x)$ has a unique minimum on $(0,1)$.

Direct calculation gives

$$g_{a,b}(0)=a+b-\frac{2}{\pi} \tag{20}$$

and

$$\lim_{x\to1^-} g_{a,b}(x)=2a-1. \tag{21}$$

Therefore,

(1) if $a-b\ge\frac{4}{\pi^2}$ and $a+b\ge\frac{2}{\pi}$, then $g_{a,b}(x)$ and $f'_{a,b}(x)$ are positive, and so the function $f_{a,b}(x)$ is increasing on $(0,1)$;

(2) if $a-b\ge\frac{4}{\pi^2}$ and $2a\le1$, then $g_{a,b}(x)$ and $f_{a,b}'(x)$ are negative, and so the function $f_{a,b}(x)$ is decreasing on $(0,1)$;
(3) if $a-b\le\frac13$ and $a+b\le\frac{2}{\pi}$, then $g_{a,b}(x)$ and $f_{a,b}'(x)$ are negative, and so the function $f_{a,b}(x)$ is decreasing on $(0,1)$;
(4) if $a-b\le\frac13$ and $2a\ge1$, then $g_{a,b}(x)$ and $f_{a,b}'(x)$ are positive, and so the function $f_{a,b}(x)$ is increasing on $(0,1)$;
(5) if $\frac13<a-b<\frac{4}{\pi^2}$, $a+b\le\frac{2}{\pi}$ and $a\le\frac12$, then $g_{a,b}(x)$ and $f_{a,b}'(x)$ are negative, and so the function $f_{a,b}(x)$ is decreasing on $(0,1)$;
(6) if $\frac13<a-b<\frac{4}{\pi^2}$, $a+b>\frac{2}{\pi}$ and $a\le\frac12$, then $g_{a,b}(x)$ and $f_{a,b}'(x)$ have a unique zero on $(0,1)$, which is a unique maximum point of $f_{a,b}(x)$ on $(0,1)$;
(7) if $\frac13<a-b<\frac{4}{\pi^2}$, $a+b\le\frac{2}{\pi}$ and $a>\frac12$, then $g_{a,b}(x)$ and $f_{a,b}'(x)$ have a unique zero on $(0,1)$, which is a unique minimum point of $f_{a,b}(x)$ on $(0,1)$;
(8) if $\frac13<a-b<\frac{4}{\pi^2}$, the minimum point $x_0\in(0,1)$ of $g_{a,b}(x)$ satisfies

$$\frac{1}{\arccos x_0}=\frac{x_0+\sqrt{x_0^2+4(a-b)(1-x_0^2)}}{2\sqrt{1-x_0^2}}$$

and the minimum of $g_{a,b}(x)$ equals

$$\begin{aligned}g_{a,b}(x_0)&=a+b+\left(a-b-\frac12\right)x_0-\frac12\sqrt{x_0^2+4(a-b)(1-x_0^2)}\\&\ge a+b-\frac{2(a-b)^{3/2}}{\sqrt{4(a-b)-1}},\end{aligned}$$

which means that
(a) when $\frac13<a-b<\frac{4}{\pi^2}$ and $a+b\ge\frac{2(a-b)^{3/2}}{\sqrt{4(a-b)-1}}$, the functions $g_{a,b}(x)$ and $f_{a,b}'(x)$ are non-negative, and so the function $f_{a,b}(x)$ is strictly increasing on $(0,1)$;
(b) when $\frac13<a-b<\frac{4}{\pi^2}$, $a+b>\frac{2}{\pi}$, $a>\frac12$ and $a+b<\frac{2(a-b)^{3/2}}{\sqrt{4(a-b)-1}}$, the functions $g_{a,b}(x)$ and $f_{a,b}'(x)$ have two zeros which are in sequence the maximum and minimum of the function $f_{a,b}(x)$ on $(0,1)$.

As a result, the sufficiency for the function $f_{a,b}(x)$ to be monotonic on $(0,1)$ is proved.

Conversely, if the function $f_{a,b}(x)$ is strictly decreasing, then the function $g_{a,b}(x)$ must be negative on $(0,1)$, so the quantities in (20) and (21) are non-positive. Hence, the condition in (3) is also necessary.

By the similar argument as above, the necessary condition (9) follows immediately. The proof of Theorem 1 is thus proved. □

Proof of Theorem 2. It is easy to see that

$$\lim_{x\to0^+}f_{a,b}(x)=\frac{\pi}{2}$$

and

$$\lim_{x\to1^-}f_{a,b}(x)=2^b\lim_{x\to1^-}\frac{\arccos x}{(1-x)^a}=\begin{cases}2^{b+1/2}, & a=\frac12;\\0, & a<\frac12;\\\infty, & a>\frac12.\end{cases}$$

From Theorem 1, it follows that the function $f_{1/2,b}(x)$ is strictly increasing (or strictly decreasing respectively) on $(0,1)$ if $b\ge\frac16$ (or if and only if $b\le\frac{2}{\pi}-\frac12$ respectively). Consequently, if $b\ge\frac16$, then

$$\frac{\pi}{2}=\lim_{x\to0^+}f_{1/2,b}(x)<f_{1/2,b}(x)<\lim_{x\to1^-}f_{1/2,b}(x)=2^{b+1/2} \tag{22}$$

on $(0,1)$, which can be rearranged as the inequality (10); if $b \le \frac{2}{\pi} - \frac12$, the inequality (22), and so the inequality (10), reverses.

The right-hand side inequality in (10) may be rewritten as

$$\begin{aligned}
b &> \frac{\ln \arccos x - \frac12 \ln(1-x) - \frac12 \ln 2}{\ln 2 - \ln(1+x)} \\
&\to (1+x)\left[\frac{1}{\sqrt{1-x^2}\,\arccos x} - \frac{1}{2(1-x)}\right] \\
&\to \frac{2(x-1) + \sqrt{1-x^2}\,\arccos x}{(x-1)\sqrt{1-x^2}\,\arccos x} \\
&\to \frac{x \arccos x/\sqrt{1-x^2}\, - 1}{(x-1)\big[1 + (1+2x)\arccos x/\sqrt{1-x^2}\,\big]} \\
&\to \frac{x \arccos x/\sqrt{1-x^2}\, - 1}{4(x-1)} \\
&\to \frac16
\end{aligned}$$

as $x \to 1^-$. Therefore, the condition $b \ge \frac16$ is also a necessary condition such that the right-hand side inequality in (10) is valid.

The reversed version of the left-hand side inequality in (10) may be rearranged as

$$b < \frac{\ln \pi - \ln 2 + [\ln(1-x)]/2 - \ln \arccos x}{\ln(1+x)} \to \frac{2}{\pi} - \frac12$$

as $x \to 0^+$. Hence, the necessity of $\frac{2}{\pi} - \frac12$ is proved.

By the equation (19) in the proof of Theorem 1, it follows that the extreme points $\xi \in (0,1)$ of the function $f_{a,b}(x)$ satisfy $g_{a,b}(\xi) = 0$, that is,

$$\arccos \xi = \frac{\sqrt{1-\xi^2}}{a+b+(a-b)\xi},$$

so the extremes of $f_{a,b}(x)$ equal

$$f_{a,b}(\xi) = \frac{(1+\xi)^{b+1/2}}{(1-\xi)^{a-1/2}[a+b+(a-b)\xi]} \triangleq g(\xi)$$

and

$$g'(x) \triangleq \frac{(x+1)^{b-1/2} h(x)}{[a+b+(a-b)x]^2 (1-x)^{a+1/2}},$$

where

$$h(x) = (a-b)^2 x^2 + (a+b)(2a-2b-1)x + (a+b)^2 - a + b$$

has two zero points x_1 and x_2 which are also the zeros of the function $g'(x)$ and the extreme points of $g(x)$ for $x \in (0,1)$.

When $16ab(b-a) + (a+b)^2 > 0$ and $x_{1,2} \in (0,1)$, the point x_1 is the maximum point and x_2 is the minimum point of $g(x)$, so we have the inequality

$$\frac{(1+x_2)^{b+1/2}(1-x_2)^{1/2-a}}{a+b+(a-b)x_2} \le f_{a,b}(\xi) \le \frac{(1+x_1)^{b+1/2}(1-x_1)^{1/2-a}}{a+b+(a-b)x_1}. \tag{23}$$

When $16ab(b-a) + (a+b)^2 > 0$ such that $x_1 \le 0$ and $x_2 \in (0,1)$, the function $g(x)$ has only one minimum and the left-hand side inequality in (23) is valid.

When $16ab(b-a) + (a+b)^2 > 0$ such that $x_1 \in (0,1)$ and $x_2 \ge 1$, the function $g(x)$ has only one maximum and the right-hand side inequality in (23) is valid.

When $16ab(b-a)+(a+b)^2>0$ such that $x_2\le 0$ or $x_1\ge 1$, the function $g(x)$ is strictly increasing on $(0,1)$; since

$$\lim_{x\to0^+}g(x)=\frac{1}{a+b}\quad\text{and}\quad \lim_{x\to1^-}g(x)=\begin{cases}2^{b+1/2}, & a=\frac12,\\ 0, & a<\frac12,\\ \infty, & a>\frac12,\end{cases}$$

we have

$$\frac{1}{a+b}\le f_{a,b}(\xi)\le\begin{cases}2^{b+1/2}, & a=\frac12,\\ 0, & a<\frac12,\\ \infty, & a>\frac12.\end{cases}\tag{24}$$

When $16ab(b-a)+(a+b)^2>0$ such that $x_1\le 0$ and $x_2\ge 1$, the function $g(x)$ is strictly decreasing on $(0,1)$, and so the inequality (24) reverses.

When $16ab(b-a)+(a+b)^2\le 0$, the function $g(x)$ is strictly increasing on $(0,1)$, and so the inequality (24) holds.

Under the condition (6),

(1) if $16ab(b-a)+(a+b)^2>0$ and $x_1>0$, then

$$\left.\begin{aligned}&\min\biggl\{2^{b+1/2},\frac{\pi}{2}\biggr\}, & a&=\frac12\\ &0, & a&<\frac12\end{aligned}\right\}\le\frac{(1+x)^b}{(1-x)^a}\arccos x \le f_{a,b}(\xi)\le\frac{(1+x_1)^{b+1/2}(1-x_1)^{1/2-a}}{a+b+(a-b)x_1};$$

(2) if either $16ab(b-a)+(a+b)^2>0$ such that $x_2\le 0$ or $x_1\ge 1$ or $16ab(b-a)+(a+b)^2\le 0$, then

$$\left.\begin{aligned}&\min\biggl\{2^{b+1/2},\frac{\pi}{2}\biggr\}, & a&=\frac12\\ &0, & a&<\frac12\end{aligned}\right\}\le\frac{(1+x)^b}{(1-x)^a}\arccos x \le f_{a,b}(\xi)\le\begin{cases}2^{b+1/2}, & a=\dfrac12,\\ 0, & a<\dfrac12.\end{cases}$$

Under the condition (7), if $16ab(b-a)+(a+b)^2>0$ and $x_2\in(0,1)$, then

$$\frac{(1+x)^b}{(1-x)^a}\arccos x\ge f_{a,b}(\xi)\ge\frac{(1+x_2)^{b+1/2}(1-x_2)^{1/2-a}}{a+b+(a-b)x_2}.$$

Straightforward simplification completes the proof of Theorem 2. □

Proof of Theorem 3. Direct computation yields

$$\begin{aligned}\frac{\mathrm{d}F_{1/2,1/2,2\sqrt2}(x)}{\mathrm{d}x}&=\frac{\bigl[1+\sqrt{2(x+1)}\,\bigr]\sqrt{1-x^2}}{(1+x)(x-1)^2}\\&\quad\times\left[\arccos x-\frac{\bigl(\sqrt{1+x}\,+2\sqrt2\,\bigr)\sqrt{1-x}}{1+\sqrt{2(x+1)}}\right]\\&\triangleq\frac{\bigl[1+\sqrt{2(x+1)}\,\bigr]\sqrt{1-x^2}}{(1+x)(x-1)^2}G(x),\end{aligned}$$

$$G'(x)=\frac{(x-1)\sqrt{2(1+x)}\,\bigl[\sqrt{1+x}\,-\sqrt2\,\bigr]}{2\sqrt{(1+x)(1-x^2)}\,\bigl[1+\sqrt{2(1+x)}\,\bigr]^2}$$

$$> 0.$$

Thus, the function $G(x)$ is strictly increasing on $(0,1)$. Since $\lim_{x\to 1^-} G(x) = 0$, the function $G(x)$ is negative on $(0,1)$, which means that the derivative $F'_{1/2,1/2,2\sqrt{2}}(x)$ is negative and that the function $F_{1/2,1/2,2\sqrt{2}}(x)$ is strictly decreasing on $(0,1)$. Further, from

$$\lim_{x\to 0^+} F_{1/2,1/2,2\sqrt{2}}(x) = \left(\frac{1}{2} + \sqrt{2}\right)\pi \quad \text{and} \quad \lim_{x\to 1^-} F_{1/2,1/2,2\sqrt{2}}(x) = 6,$$

the double inequality (16) and its best possibility follow. □

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(B.-N. Guo) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com

URL: http://guobaini.spaces.live.com

(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com

URL: http://qifeng618.spaces.live.com